ALGEBRAIC CONNECTIVITY OF CONNECTED GRAPHS WITH FIXED NUMBER OF PENDANT VERTICES

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Abstract. In this paper we consider the following problem: Over the class of all simple connected graphs of order n with k pendant vertices (n, k being fixed), which graph maximizes (respectively, minimizes) the algebraic connectivity? We also discuss the algebraic connectivity of unicyclic graphs.

Keywords: Laplacian matrix; Algebraic connectivity; Characteristic set; Perron component; Pendant vertex.

1. Introduction

All graphs considered here are finite, simple and undirected. Let G be a graph with vertex set $V = \{v_1, \cdots, v_n\}$. The number n of vertices is called the *order* of G. The *adjacency matrix* $A(G)$ of G is defined as $A(G) = [a_{ij}]$, where a_{ij} is equal to one, if the unordered pair $\{v_i, v_j\}$ is an edge of G and zero, otherwise. Let $D(G)$ be the diagonal matrix of the vertex degrees of G. The *Laplacian matrix* $L(G)$ of G is defined as $L(G) = D(G) - A(G)$. We refer to [13, 14] for a general overview on results related to Laplacians. It is well known that $L(G)$ is a symmetric positive semidefinite M-matrix. The smallest eigenvalue of $L(G)$ is zero with the vector of all ones as its eigenvector. It has multiplicity one if and only if G is connected. In other words, the second smallest eigenvalue of $L(G)$ is positive if and only if G is connected. Viewing the second smallest eigenvalue as an algebraic measure of connectivity, Fiedler termed this eigenvalue as the *algebraic connectivity* of G, denoted $\mu(G)$. The following two lemmas are well known.

Lemma 1.1 ([4], p.223). *Let G be a graph. Let $\widehat{G}$ be the graph obtained from G by adding a pendant vertex to a vertex of G. Then $\mu(\widehat{G}) \leq \mu(G)$.*

Lemma 1.2 ([5], 3.2, p.299). *Let G be a non-complete graph. Let G' be the graph obtained from G by joining two non-adjacent vertices of G with an edge. Then $\mu(G) \leq \mu(G')$.*

Lemma 1.2 implies that, over all connected graphs, the maximum algebraic connectivity occurs for the complete graph and the minimum algebraic connectivity occurs for a tree. It is also known that over all trees of order n, the path (denoted by P_n) has the minimum

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algebraic connectivity and this minimum is given by $2(1 - \cos \frac{\pi}{n})$ ([5], p.304). Merris proved that, among all trees of order $n > 2$, the star $K_{1,n-1}$ uniquely attains the maximum algebraic connectivity which is equal to 1 ([12], Corollary 2, p.118).

There is a good deal of work on the algebraic connectivity of graphs. We refer to [1]–[7],[10],[17] for various works on the algebraic connectivity of connected graphs having certain graph theoretic properties. For all graph theoretic terms used in this paper (but not defined), the reader is advised to look at the book Graph Theory by Harary [9]. In this paper, we consider the problem of extremizing the algebraic connectivity over all connected graphs of order n with k pendant vertices for fixed n and k, $0 \leq k \leq n-1$.

The paper is arranged as follows: In Section 2, we recall some results from the literature that will be used in the subsequent sections. In Section 3, we record some elementary results and prove a result which gives conditions under which equality is attained in Lemma 1.2. We study the algebraic connectivity of connected graphs with and without pendant vertices in Section 4 and Section 5, respectively. Finally, in Section 6, we discuss the algebraic connectivity of unicyclic graphs.

2. Preliminaries

Let G be a connected graph. The *distance* $d(u,v)$ between two vertices u and v of G is the length of a shortest path from u to v. The *diameter* of G is defined by $\max\limits_{u,v} d(u,v)$. A *pendant vertex* of G is a vertex of degree one. For a set W of vertices of G, $G - W$ denotes the graph obtained from G by removing the vertices in W and all edges incident with them. If $W = \{v\}$ consists of one vertex only, we simply denote $G - W$ by $G - v$ and refer to the connected components of $G - v$ as the *connected components of* G *at* v. A vertex v of G is called a *cut-vertex* if there are at least two connected components of G at v.

An eigenvector of $L(G)$ corresponding to $\mu(G)$ is called a *Fiedler vector* of G. Let Y be a Fiedler vector of G. By $Y(v)$, we mean the co-ordinate of Y corresponding to the vertex v. A vertex v of G is called a *characteristic vertex* with respect to Y if $Y(v) = 0$ and there exists a vertex w adjacent to v such that $Y(w) \neq 0$. An edge $\{u,w\}$ of G is called a *characteristic edge* with respect to Y if $Y(u)Y(w) < 0$. The *characteristic set* of G with respect to Y, denoted $\mathcal{C}(G,Y)$, is the set of all characteristic vertices and edges of G. By a result of Fiedler [6], either $\mathcal{C}(G,Y)$ consists of one vertex only or $\mathcal{C}(G,Y)$ is contained in a block of G. For a tree, it is known that $\mathcal{C}(G,Y)$ is independent of Y and it contains either a vertex or an edge. We refer to [1] for more general results on the size of $\mathcal{C}(G,Y)$.

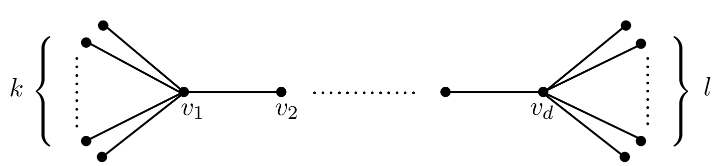

Figure 1. The tree $T(k,l,d)$

For a fixed positive integer n, the path $[v_1v_2\cdots v_n]$ on n vertices is denoted by P_n. For positive integers k, l, d with $n = k + l + d$, let $T(k, l, d)$ be the tree of order n obtained by taking the path P_d and adding k pendant vertices adjacent to v_1 and l pendant vertices adjacent to v_d (see Figure 1). The path Note that $T(1, 1, d]$ is a path on $d+2$ vertices. The next proposition determines the tree, up to isomorphism, which minimizes the algebraic connectivity over all trees of order n with fixed diameter.

Proposition 2.1 ([2], Theorem 3.2, p.58). *Among all trees of order n with fixed diameter $d+1$, the minimum algebraic connectivity is uniquely attained by $T\left(\lceil\frac{n-d}{2}\rceil, \lfloor\frac{n-d}{2}\rfloor, d\right)$.*

Now, consider a path $v_1v_2\cdots v_d$ on $d \geq 3$ vertices and add $n-d$ pendant vertices adjacent to the vertex v_j, where $j = \lfloor\frac{d+1}{2}\rfloor$. The new graph thus constructed is denoted by T^d_{n-d} (see Figure 2). Then T^d_{n-d} is a tree of order n with diameter $d-1$. The next proposition determines the tree, up to isomorphism, which maximizes the algebraic connectivity over all trees of order n with fixed diameter.

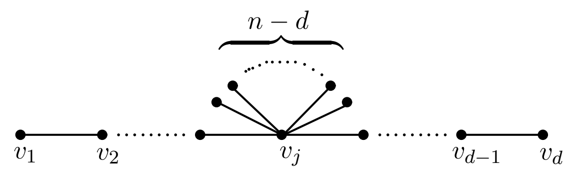

FIGURE 2. The tree T^d_{n-d}

Proposition 2.2 ([2], p.62–65). *Among all trees of order n with fixed diameter $d+1$, the maximum algebraic connectivity is uniquely attained by T^{d+2}_{n-d-2}.*

Fix a vertex v of G and let $l, k \geq 1$. We construct a new graph $G_{k,l}$ from G by attaching two new paths $P : vv_1v_2\dots v_k$ and $Q : vu_1u_2\dots u_l$ at v of lengths k and l, respectively, where $u_1, u_2, \dots, u_l, v_1, v_2, \dots, v_k$ are distinct new vertices. Let $\widetilde{G}_{k,l}$ be the graph obtained from $G_{k,l}$ by removing the edge $\{v_{k-1}, v_k\}$ and adding a new edge $\{u_l, v_k\}$ (see Figure 3). We say that $\widetilde{G}_{k,l}$ is obtained from $G_{k,l}$ by *grafting* an edge. The next result compares the algebraic connectivity of $G_{k,l}$ with that of $\widetilde{G}_{k,l} \simeq G_{k-1,l+1}$ whenever the initial graph G is a tree.

Proposition 2.3 ([17], Theorem 2.4, p.861). *Let G be a tree of order $n \geq 2$ and v be a vertex of G. For $l, k \geq 1$, let $G_{k,l}$ and $\widetilde{G}_{k,l} \simeq G_{k-1,l+1}$ be the graphs as defined above with respect to v. If $l \geq k$, then $\mu(G_{k-1,l+1}) \leq \mu(G_{k,l})$.*

Let v be a vertex of G and $C_1, C_2, \cdots, C_k$ be the connected components of G at v. Note that $k \geq 2$ if and only if v is a cut-vertex of G. For each $i = 1, \cdots, k$, we denote by $\widehat{L}(C_i)$ the principal sub-matrix of $L(G)$ corresponding to the vertices of C_i. Since $L(G)$ is an M-matrix and has nullity 1, it follows that $\widehat{L}(C_i)$ is a non-singular M-matrix and hence $\widehat{L}(C_i)^{-1}$ is a positive matrix, called the *bottleneck matrix* of C_i. By the Perron-Frobenius Theorem [15], $\widehat{L}(C_i)^{-1}$ has a simple positive dominant eigenvalue, called the *Perron value*

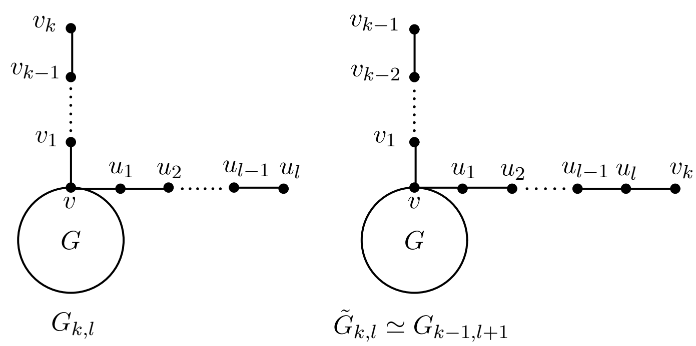

FIGURE 3. Grafting an edge

of C_i and is denoted by $\rho(\widehat{L}(C_i)^{-1})$. A corresponding eigenvector of $\rho(\widehat{L}(C_i)^{-1})$ with all entries positive is called a *Perron vector of* C_i. We say that C_i is a *Perron component of* G *at* v if its Perron value is maximum among that the Perron values of $C_1, \cdots, C_k$.

The next proposition gives the description of the entries of the bottleneck matrices of a tree.

Proposition 2.4 ([11], Proposition 1, p.313)**.** *Let T be a tree, v be a vertex of T and C be a connected component of T at v. Then $\widehat{L}(C)^{-1} = [m_{ij}]$, where m_{ij} is the number of common edges between the paths P_{iv} joining i and v and P_{jv} joining j and v.*

The following proposition connecting Perron components, bottleneck matrices and algebraic connectivity recasts some of the results obtained in [2, 10]. Throughout the paper, we denote by I the identity matrix and by J the matrix of all ones of appropriate orders. For a symmetric matrix M, $\lambda(M)$ denotes its largest eigenvalue.

Proposition 2.5. *Let G be a connected graph with a cut-vertex v. Let $C_1, C_2, \cdots, C_k$ be the connected components of G at v with C_1 as a Perron component. Then the following results hold.*

(i) *There is a unique non-negative number x such that*

$$\lambda\left(\widehat{L}(C_1)^{-1} - xJ\right) = \rho\left(\widehat{L}(C_2)^{-1} \oplus \cdots \oplus \widehat{L}(C_k)^{-1} \oplus [0] + xJ\right) = \frac{1}{\mu(G)}.$$

(ii) *If there exists a non-negative number x such that*

$$\lambda\left(\widehat{L}(C_1)^{-1} - xJ\right) = \rho\left(\widehat{L}(C_2)^{-1} \oplus \cdots \oplus \widehat{L}(C_k)^{-1} \oplus [0] + xJ\right) = \frac{1}{\alpha},$$

then α is an eigenvalue of $L(G)$.

The next result is a special case of Proposition 2.5(i).

Proposition 2.6 ([10], Theorem 8, p.146)**.** *Let G be a connected graph and $\{v_i, v_j\}$ be an edge not on any cycle of G. Let C_i be the connected component of G at v_j containing v_i and C_j be the connected component of G at v_i containing v_j. Then C_i is the Perron*

component of G at v_j and C_j is the Perron component of G at v_i if and only if there exists a $\gamma \in (0,1)$ such that

$$\rho\left(\widehat{L}(C_i)^{-1} - \gamma J\right) = \rho\left(\widehat{L}(C_j)^{-1} - (1-\gamma)J\right) = \frac{1}{\mu(G)}.$$

In that case, $\mu(G)$ is a simple eigenvalue of $L(G)$ and the characteristic set of G is a singleton consisting of the edge $\{v_i, v_j\}$.

Identification of Perron components at a vertex helps to determine the location of the characteristic set in G. The next proposition is a result in that direction.

Proposition 2.7 ([10], Theorem 1 and Corollary 1.1)**.** *Let G be a connected graph. For any vertex v of G which is not a characteristic vertex, the unique Perron component at v contains the vertices in any characteristic set. A cut-vertex v is a characteristic vertex of G if and only if there are at least two Perron components of G at v, and in that case*

$$\mu(G) = \frac{1}{\rho(\widehat{L}(C)^{-1})},$$

where C is a Perron component of G at v.

Remark 2.8. *Recall that P_{2n+1} is a path on $2n+1$ vertices with the vertex v_i being adjacent to the vertex v_{i+1} for $i = 1, 2, \ldots, 2n$. Let C_1 and C_2 be the two components of $P_{2n+1} - v_{n+1}$. Then using symmetry, it can be easily observed that the vertex labelled v_{n+1} is the characteristic vertex. Hence by Proposition 2.7*

$$\rho(\widehat{L}(C)^{-1}) = \frac{1}{\mu(P_{2n+1})} = \frac{1}{2(1-\cos(\frac{\pi}{2n+1}))}.$$

For non-negative square matrices A and B (not necessarily of the same order), $A \ll B$ means that there exists a permutation matrix P such that PAP^T is entri-wise dominated by a principal sub-matrix of B, with strict inequality in at least one position in the case A and B have the same order. A useful fact from the Perron-Frobenius theory is that if B is irreducible and $A \ll B$, then $\rho(A) < \rho(B)$. We use this fact together with Propositions 2.4 and 2.5 (mostly without mention) to get information about the algebraic connectivity of a graph, and in particular, for a tree.

3. Initial Results

We start with the following observation. Let K_n be the complete graph of order $n \geq 2$ and v be a vertex of K_n. Let C denote the only connected component of K_n at v. We have $L(K_n) = nI - J$, an n-by-n matrix. Let $\widehat{L}(C)$ be the principal sub-matrix of $L(K_n)$ corresponding to the vertices of C. Then $\widehat{L}(C) = nI - J$, an $(n-1)$-by-$(n-1)$ matrix. It can be verified that $\widehat{L}(C)^{-1} = \frac{1}{n}I + \frac{1}{n}J$. Since the sum of each row of $\widehat{L}(C)^{-1}$ is one, the vector of all ones is an eigenvector of $\widehat{L}(C)^{-1}$ corresponding to the eigenvalue one. So, by the Perron-Frobenius Theorem, the Perron value of C is one.

Lemma 3.1. *Let v be a vertex of a connected graph G. For a connected component C of G at v, let $\widetilde{C}_v$ denote the induced subgraph of G on the vertices of C together with v. Then the following results hold.*

(i) *If $\widetilde{C}_v$ is complete, then the Perron value of C is one.*

(ii) *If v is a cut-vertex and $\widetilde{C}_v$ is complete for each connected component C of G at v, then $\mu(G) = 1$.*

Proof. (i) The principal sub-matrix of $L(G)$ and that of $L(\widetilde{C}_v)$ corresponding to the vertices of C are the same. If $\widetilde{C}_v$ is complete, then replacing K_n by $\widetilde{C}_v$ in the first paragraph of this section, it follows that the Perron value of C is one.

(ii) By (i), $\rho(\widehat{L}(C)^{-1}) = 1$ for each connected component C of G at v as $\mu(G) = 1$ follows from Proposition 2.7. □

A consequence of Lemma 3.1 is the following.

Corollary 3.2. *Let G be a connected graph with a cut-vertex v. Then $\mu(G) \leq 1$.*

Proof. Let $\widehat{G}$ be the graph obtained from G by making $\widetilde{C}_v$ complete for each connected component C of G at v. By Lemma 1.2, $\mu(G) \leq \mu(\widehat{G})$. Applying Lemma 3.1$(ii)$ to the pair $(\widehat{G}, v)$, we get that $\mu(\widehat{G}) = 1$. So $\mu(G) \leq 1$. □

The statements of Lemma 3.1 and Corollary 3.2 were implicitly mentioned in ([2], Example 1.5, p.51). We write them here with their proofs for the sake of completeness and for later use in this paper. We also note that Corollary 3.2 follows from ([5], 4.1, p.303) since the vertex connectivity of G is one.

Lemma 3.3. *Let v be a vertex of a connected graph G. If v is not a cut-vertex of G, then the Perron value of $C = G - v$ (the only connected component of G at v) is at least one.*

Proof. Let $\widetilde{G}$ be the graph obtained from G by adding one pendant vertex adjacent to v. Let $C_1 = C$ and C_2 be the two connected components of $\widetilde{G}$ at v. The principal sub-matrix of $L(G)$ and that of $L(\widetilde{G})$ corresponding to the vertices of C_1 are the same. The Perron value of C_2 is 1 and $\mu(\widetilde{G}) \leq 1$ (Corollary 3.2). Then the component C_2 cannot be a Perron component (use Proposition 2.5) and hence $\rho(\widehat{L}(C)^{-1}) = \rho(\widehat{L}(C_1)^{-1}) \geq \rho(\widehat{L}(C_2)^{-1}) = 1$. □

The next theorem gives conditions under which the equality is attained in Lemma 1.2.

Theorem 3.4. *Let v be a vertex of a connected graph G such that v is not a cut-vertex of G. Form a new graph $\widetilde{G}$ from G by adding $t \geq 1$ pendant vertices $v_1, \cdots, v_t$ adjacent to v. Let $\widehat{G}$ be the graph obtained from $\widetilde{G}$ by adding k, $0 \leq k \leq \frac{t(t-1)}{2}$, edges among $v_1, \cdots, v_t$. Then $\mu(\widetilde{G}) = \mu(\widehat{G})$.*

Proof. Let G^* be the graph obtained from $\widetilde{G}$ by adding $\frac{t(t-1)}{2}$ edges among $v_1, \cdots, v_t$. By Lemma 1.2 and Corollary 3.2, $\mu(\widetilde{G}) \leq \mu(\widehat{G}) \leq \mu(G^*) \leq 1$. If $\mu(\widetilde{G}) = 1$, we are done. So, we assume that $\mu(\widetilde{G}) < 1$ and prove that $\mu(G^*) \leq \mu(\widetilde{G})$.

Let $C_1, C_2, \cdots, C_{t+1}$ be the $t+1$ connected components of $\widetilde{G}$ at v, where $C_1 = G - v$. For $2 \le i \le t+1$, the Perron value of C_i is one. As v is not a cut-vertex of G, by Lemma 3.3, the Perron value of C_1 is at least 1. So, by definition, C_1 is a Perron component of $\widetilde{G}$ at v. By Proposition 2.5(i), there is a unique non-negative number θ such that

$$\lambda\left(\widehat{L}(C_1)^{-1} - \theta J\right) = \rho\left(\widehat{L}(C_2)^{-1} \oplus \cdots \oplus \widehat{L}(C_{t+1})^{-1} \oplus [0] + \theta J\right) = \frac{1}{\mu(\widetilde{G})}.$$

Since $\mu(\widetilde{G}) < 1$, the second equality implies that $\theta > 0$.

Let $M = \widehat{L}(C_2)^{-1} \oplus \cdots \oplus \widehat{L}(C_{t+1})^{-1} \oplus [0] + \theta J$. Thus M is a $(t+1)$-by-$(t+1)$ positive matrix with Perron value $\frac{1}{\mu(\widetilde{G})}$. Let $Y = [y_1, \cdots, y_t, y_{t+1}]^T$ be a Perron vector of M and take $\beta = \theta(y_1 + \cdots + y_{t+1})$. Then $\beta > 0$. A simple calculation shows that $y_i + \beta = \frac{1}{\mu(\widetilde{G})} y_i$ for $1 \le i \le t$ and $\beta = \dfrac{1}{\mu(\widetilde{G})} y_{t+1}$. Since $0 < \mu(\widetilde{G}) < 1$, it follows that $y_1 = \cdots = y_t$. So without loss of generality we may take $Y = [1, \cdots, 1, y]^T$ for some $y > 0$.

There are two connected components of G^* at v, say $D_1 = C_1$ and D_2. The Perron value of D_2 is one by Lemma 3.1(i). So D_1 is a Perron component of G^* at v since it has Perron value at least one. Let $M^* = \widehat{L}(D_2)^{-1} \oplus [0] + \theta J$. Using $\widehat{L}(D_2)^{-1} = \frac{1}{t+1} I + \frac{1}{t+1} J$ (see the first paragraph of this section), it can be verified that $M^* Y = \frac{1}{\mu(\widetilde{G})} Y$. So, by the Perron-Frobenius Theorem, $\rho(M^*) = \frac{1}{\mu(\widetilde{G})}$. Thus,

$$\rho\left(\widehat{L}(D_2)^{-1} \oplus [0] + \theta J\right) = \frac{1}{\mu(\widetilde{G})} = \lambda\left(\widehat{L}(C_1)^{-1} - \theta J\right) = \lambda\left(\widehat{L}(D_1)^{-1} - \theta J\right).$$

Now, by Proposition 2.5(ii), $\mu(\widetilde{G})$ is an eigenvalue of $L(G^*)$ and hence $\mu(G^*) \le \mu(\widetilde{G})$ as, by definition, $\mu(G^*)$ is the smallest nonzero eigenvalue of $L(G^*)$. This completes the proof. □

4. With Pendant Vertices

Let $\mathfrak{H}_{n,k}$ denote the class of all connected graphs of order n with k pendant vertices. We may assume that $1 \le k \le n-1$ and hence $n \ge 3$. We first consider the question of maximizing the algebraic connectivity over $\mathfrak{H}_{n,k}$. We now define a collection of graphs, denoted P_n^k, which have exactly k pendant vertices and hence are members of $\mathfrak{H}_{n,k}$. For $k \ne n-2$, the graph P_n^k is obtained by adding k pendant vertices adjacent to a single vertex of the complete graph K_{n-k}. There is no graph of order $n = 3$ with exactly one pendant vertex. For $n \ge 4$ and $k = n-2$, the graph P_n^{n-2} is obtained by adding $n-3$ pendant vertices adjacent to a pendant vertex of the path P_3.

Theorem 4.1. *The graph P_n^k, $k \ne n-2$, attains the maximum algebraic connectivity over $\mathfrak{H}_{n,k}$ and this maximum value is equal to one.*

Proof. Let G be a graph in $\mathfrak{H}_{n,k}$. Since $k \ge 1$, G has a cut-vertex. By Corollary 3.2, $\mu(G) \le 1$. Also P_n^k, $k \ne n-2$, has a cut-vertex satisfying the hypothesis of Lemma 3.1(ii). So $\mu(P_n^k) = 1$ and hence $\mu(G) \le \mu(P_n^k)$. □

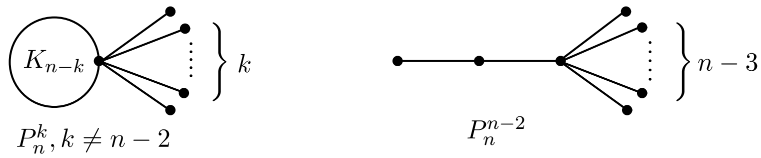

FIGURE 4. The graph P_n^k

Remark 4.2 (Uniqueness). *For $k = n-1$, $\mathfrak{H}_{n,k}$ contains only one graph, namely, $K_{1,n-1} \simeq P_n^{n-1}$. For $k = n-3$, let G be a graph in $\mathfrak{H}_{n,n-3}$. If G is a tree then clearly $\mu(G) < 1$. Otherwise G has a unique 3-cycle. By ([2], Theorem 4.14, p.73), it follows that $\mu(G) = 1$ if and only if $G \simeq P_n^{n-3}$. However, for $1 \leq k \leq n-4$, P_n^k is not the unique graph (up to isomorphism) in $\mathfrak{H}_{n,k}$ having algebraic connectivity one. We give an example below.*

Example 4.3. *Let $1 \leq k \leq n-4$. Let $v_1, \cdots, v_{n-k-1}$ be $n-k-1$ pendant vertices in the star $K_{1,n-1}$. Let G be the graph obtained from $K_{1,n-1}$ by adding new edges $\{v_i, v_{i+1}\}$, $1 \leq i \leq n-k-2$. Then G is a member of $\mathfrak{H}_{n,k}$. Since $n \geq 5$, G is not isomorphic to P_n^k. But by Theorem 3.4, $\mu(G) = \mu(K_{1,n-1}) = 1$.*

Theorem 4.4. *The graph P_n^{n-2} uniquely maximizes the algebraic connectivity over $\mathfrak{H}_{n,n-2}$.*

Proof. Any graph in $\mathfrak{H}_{n,n-2}$ is a tree with diameter three. By Proposition 2.2, the maximum algebraic connectivity is uniquely attained by the tree $T^4_{n-4} \simeq P_n^{n-2}$ over $\mathfrak{H}_{n,n-2}$. □

We next consider the question of minimizing the algebraic connectivity over $\mathfrak{H}_{n,k}$. We first prove the result for $k = 1$ (and so $n \geq 4$). We denote by C_3^{n-3} the graph obtained by adding a path of order $n-3$ to a vertex of K_3 (see Figure 5). Then C_3^{n-3} is a member of $\mathfrak{H}_{n,1}$.

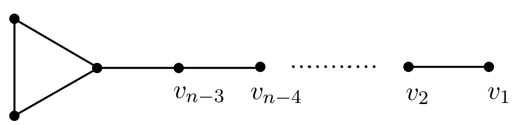

FIGURE 5. The graph C_3^{n-3}

Theorem 4.5. *The graph C_3^{n-3} uniquely attains the minimum algebraic connectivity over $\mathfrak{H}_{n,1}$.*

Proof. For $n = 4$, C_3^1 is the only graph in $\mathfrak{H}_{4,1}$. For $n = 5$, C_4^1 (adding a pendant vertex adjacent to a vertex of a 4-cycle) is the only graph in $\mathfrak{H}_{5,1}$, non-isomorphic to C_3^2, with least possible edges. The other graphs, non-isomorphic to C_3^2, in $\mathfrak{H}_{5,1}$ can be obtained from C_4^1 by joining some non-adjacent vertices. So, for any other graph G on 5 vertices and with one pendant vertex,

$$\mu(G) \geq \mu(C_4^1) \approx 0.8299 > 0.5188 \approx \mu(C_3^2).$$

Let $n \geq 6$. Any graph in $\mathfrak{H}_{n,1}$ has at least one cycle. Let G be a graph in $\mathfrak{H}_{n,1}$ which is not isomorphic to C_3^{n-3}. Delete some of the edges from the cycles of G to get a tree G_1. Further, we delete these edges in such a way that G_1 is neither isomorphic to the path P_n nor to the graph G_2 (G is not isomorphic to C_3^{n-3}) as shown in Figure 6. This is possible

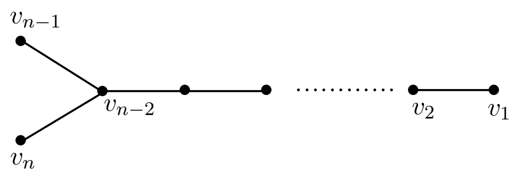

FIGURE 6. The tree G_2

since $n \geq 6$. By Lemma 1.2, $\mu(G_1) \leq \mu(G)$. Starting with G_1, by a finite sequence of graph operations consisting of grafting of edges, we can get the tree G_2 from G_1. Also note that we must get the graph $\widehat{G}_2$ as shown in Figure 7 in the penultimate step of getting G_2. Since $G_2 \simeq T(2, 1, n-3)$, $\mu(G_2) < \mu(\widehat{G}_2)$ by Proposition 2.1 and $\mu(\widehat{G}_2) \leq \mu(G_1)$ by

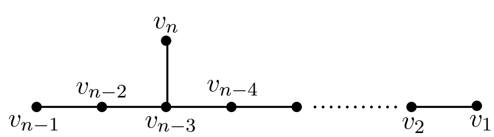

FIGURE 7. The tree $\widehat{G}_2$

Proposition 2.3. Now, the graph C_3^{n-3} is obtained from G_2 by adding the edge $\{v_{n-1}, v_n\}$. By Theorem 3.4, $\mu(C_3^{n-3}) = \mu(G_2)$ and it follows that $\mu\left(C_3^{n-3}\right) < \mu(G)$. This completes the proof. ☐

We now consider the case $k \geq 2$. Recall that the tree $T\left(\lceil \frac{k}{2} \rceil, \lfloor \frac{k}{2} \rfloor, n-k\right)$ of order n has k pendant vertices and diameter $n-k+1$.

Theorem 4.6. *The tree $T\left(\lceil \frac{k}{2} \rceil, \lfloor \frac{k}{2} \rfloor, n-k\right)$ uniquely attains the minimum algebraic connectivity over $\mathfrak{H}_{n,k}$, $k \geq 2$.*

Proof. The result is obvious for $k = 2$. Let $k \geq 3$ and G be a graph in $\mathfrak{H}_{n,k}$ which is not isomorphic to $T\left(\lceil \frac{k}{2} \rceil, \lfloor \frac{k}{2} \rfloor, n-k\right)$. If G is a tree then the result follows from Proposition 2.1. So, let us assume that G is not a tree. Now, carefully remove some of the edges from the cycles of G, to get a tree $\widehat{G}$ which is non-isomorphic to $T\left(\lceil \frac{k}{2} \rceil, \lfloor \frac{k}{2} \rfloor, n-k\right)$. Since $\widehat{G}$ has at least k pendant vertices, it has diameter at most $n-k+1$. If the diameter of $\widehat{G}$ is strictly less than $n-k+1$, then form a new tree $\widetilde{G}$ of diameter $n-k+1$ from $\widehat{G}$ by grafting of edges such that $\widetilde{G}$ is not isomorphic to $T\left(\lceil \frac{k}{2} \rceil, \lfloor \frac{k}{2} \rfloor, n-k\right)$. Now, Lemma 1.2 and Propositions 2.1 and 2.3 complete the proof. ☐

Let $\mathfrak{T}_{n,k}$ denote the subclass of $\mathfrak{H}_{n,k}$ consisting of all trees of order n with k, $2 \leq k \leq n-1$, pendant vertices. By Theorem 4.6, the tree $T(\lceil \frac{k}{2} \rceil, \lfloor \frac{k}{2} \rfloor, n-k)$ has the minimum algebraic

connectivity over $\mathfrak{T}_{n,k}$. Now the following natural question arises: Which tree attains the maximum algebraic connectivity over $\mathfrak{T}_{n,k}$? We answer this question in Theorem 4.7 below.

Let $q = \lfloor \frac{n-1}{k} \rfloor$ and $n-1 = kq+r,\ 0 \le r \le k-1$. Let $T_{n,k}$ be the tree obtained by adding r paths each of length q and $k-r$ paths each of length $q-1$ to a given vertex. Then $T_{n,k}$ is a member of $\mathfrak{T}_{n,k}$.

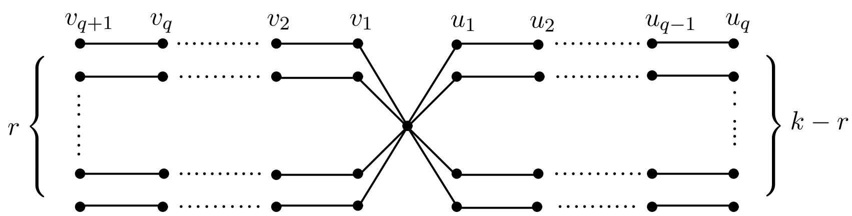

FIGURE 8. The tree $T_{n,k}$

Theorem 4.7. *The tree $T_{n,k}$ uniquely attains the maximum algebraic connectivity over $\mathfrak{T}_{n,k}$.*

Proof. For $k=2$, $T_{n,2} \simeq P_n$ is the only element in $\mathfrak{T}_{n,2}$ and the result is obvious. Assume that $k \ge 3$. Let v be the vertex of $T_{n,k}$ of degree k. There are k connected components of $T_{n,k}$ at v, and r of them are paths of order $q+1$ and the rest $k-r$ components are paths of order q. Let d_k be the diameter of $T_{n,k}$. Then

$$d_k = \begin{cases} 2q & \text{if } r=0 \\ 2q+1 & \text{if } r=1 \\ 2q+2 & \text{if } 2 \le r \le k-1 \end{cases}.$$

Any tree in $\mathfrak{T}_{n,k}$ has diameter at least d_k, and equality holds if and only if the tree is isomorphic to $T_{n,k}$.

If $r=0$ then, by Proposition 2.7, each connected component of $T_{n,k}$ at v is a Perron component consisting of paths of order q and therefore by Remark 2.8, $\mu(T_{n,k}) = 2\left(1-\cos\frac{\pi}{2q+1}\right)$. If $r \ge 1$, then from Proposition 2.4 together with the Perron-Frobenius theory, it follows that the Perron components of $T_{n,k}$ at v are the ones with $q+1$ vertices. So $\mu(T_{n,k}) \ge 2\left(1-\cos\frac{\pi}{2q+3}\right)$ with strict inequality if $r=1$. Note that if T is any tree of order n with diameter d, then $\mu(T) \le 2\left(1-\cos\frac{\pi}{d+1}\right)$ ([8], Corollary 4.4, p.234). Thus, for any tree T in $\mathfrak{T}_{n,k}$ with diameter $\beta \ge d_k+1$, we have

$$\mu(T) \le 2\left(1-\cos\frac{\pi}{\beta+1}\right) \le 2\left(1-\cos\frac{\pi}{d_k+1}\right) \le \mu(T_{n,k}).$$

Since the second inequality is strict if $r=0$ and $r \ge 2$, and the last inequality is strict if $r=1$, the uniqueness of $T_{n,k}$ having the maximum algebraic connectivity over $\mathfrak{T}_{n,k}$ follows. This completes the proof. □

5. Without Pendant Vertex

Let $\mathcal{F}_n, n \geq 3$, be the class of all connected graphs of order n without any pendant vertex. The complete graph K_n is a member of $\mathcal{F}_n$ and it has the maximum algebraic connectivity, namely $\mu(K_n) = n$. Here, our aim is to find the graph that has the minimum algebraic connectivity over $\mathcal{F}_n$. By Lemma 1.2, we consider only those graphs in $\mathcal{F}_n$ with minimum possible edges (if we delete any edge from such a graph, then the new graph is either disconnected or has atleast one pendant vertex). Note that any two cycles in such a graph are edge disjoint. For $n = 3, 4$, the cycle C_n of order n is the only such graph. In [5], Fiedler showed that $\mu(C_m) = 2\left(1 - \cos\frac{2\pi}{m}\right)$. So $\mu(C_3) = 3$ and $\mu(C_4) = 2$. For $n = 5$, there are two such graphs G_1 and G_2 in $\mathcal{F}_5$, up to isomorphism, where G_1 is C_5 and G_2 is the graph with two 3-cycles having one common vertex. By Lemma 3.1(ii), $\mu(G_2) = 1$ and hence,

$$\mu(G_1) = 2\left(1 - \cos\frac{2\pi}{5}\right) > 1 = \mu(G_2).$$

We next consider when $n \geq 6$. Let $C_{\lfloor\frac{n}{2}\rfloor,\lceil\frac{n}{2}\rceil}$ be the graph with two cycles $C_{\lfloor\frac{n+1}{2}\rfloor}$ and $C_{\lceil\frac{n+1}{2}\rceil}$ with exactly one common vertex. Then $C_{\lfloor\frac{n}{2}\rfloor,\lceil\frac{n}{2}\rceil}$ is a member of $\mathcal{F}_n$. We first prove the following.

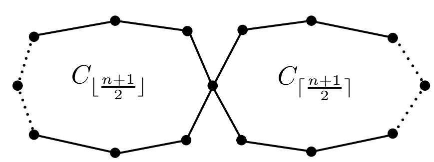

Figure 9. The graph $C_{\lfloor\frac{n}{2}\rfloor,\lceil\frac{n}{2}\rceil}$

Lemma 5.1. *For $n \geq 6$, $\mu\left(C_{\lfloor\frac{n}{2}\rfloor,\lceil\frac{n}{2}\rceil}\right) < \mu(C_n)$.*

Proof. Let v be the cut-vertex of $C_{\lfloor\frac{n}{2}\rfloor,\lceil\frac{n}{2}\rceil}$. Let C_1 and C_2 be the two connected components of $C_{\lfloor\frac{n}{2}\rfloor,\lceil\frac{n}{2}\rceil}$ at v of orders $\lfloor\frac{n-1}{2}\rfloor$ and $\lceil\frac{n-1}{2}\rceil$, respectively. For $i = 1, 2$, observe that

$$\widehat{L}(C_i) = \begin{bmatrix} 2 & -1 & 0 & \cdots & 0 & 0 \\ -1 & 2 & -1 & 0 & \cdots & 0 \\ 0 & -1 & 2 & -1 & \cdots & \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & -1 & 2 & -1 \\ 0 & \cdots & 0 & 0 & -1 & 2 \end{bmatrix}.$$

It is well known that the smallest eigenvalue of $\widehat{L}(C_1)$ is $2\left(1 - \cos\frac{\pi}{\lfloor\frac{n+1}{2}\rfloor}\right)$ and that of $\widehat{L}(C_2)$ is $2\left(1 - \cos\frac{\pi}{\lceil\frac{n+1}{2}\rceil}\right)$. If n is odd, then $\rho(\widehat{L}(C_1)^{-1}) = \rho(\widehat{L}(C_2)^{-1})$ and hence C_1 and

C_2 both are Perron components of $C_{\lfloor \frac{n}{2} \rfloor, \lceil \frac{n}{2} \rceil}$ at v. So by Proposition 2.7,

$$\mu\left(C_{\lfloor \frac{n}{2} \rfloor, \lceil \frac{n}{2} \rceil}\right) = 2\left(1 - \cos\frac{2\pi}{n+1}\right) < 2\left(1 - \cos\frac{2\pi}{n}\right) = \mu(C_n).$$

If n is even, then C_2 is the only Perron component of $C_{\lfloor \frac{n}{2} \rfloor, \lceil \frac{n}{2} \rceil}$ at v and it follows that

$$2\left(1 - \cos\frac{2\pi}{n+2}\right) < \mu\left(C_{\lfloor \frac{n}{2} \rfloor, \lceil \frac{n}{2} \rceil}\right) < 2\left(1 - \cos\frac{2\pi}{n}\right) = \mu(C_n).$$

This completes the proof. □

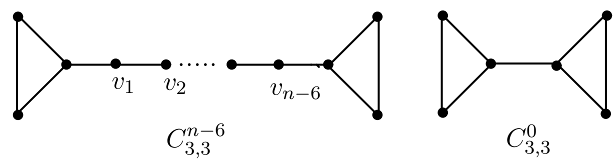

FIGURE 10. The graph $C_{3,3}^{n-6}$

Now, take a path of order $n-6$ and add a 3-cycle to each of its two pendant vertices. The new graph thus obtained is a member of $\mathcal{F}_n$. We denote it by $C_{3,3}^{n-6}$.

Theorem 5.2. *For $n \geq 6$, the graph $C_{3,3}^{n-6}$ uniquely attains the minimum algebraic connectivity over $\mathcal{F}_n$.*

Proof. In $\mathcal{F}_n$, C_n is the only graph whose maximum vertex degree is two. Since we are minimizing the algebraic connectivity over $\mathcal{F}_n$, by Lemma 5.1 we may consider graphs with maximum vertex degree at least three. Let G_1 be such a graph in $\mathcal{F}_n$ not isomorphic to $C_{3,3}^{n-6}$. Then G_1 has at least two cycles. Form a tree G_2 by deleting one edge from each cycle of G_1 in such a way that G_2 has diameter at most $n-3$ and if equality holds, then G_2 is not isomorphic to $T(2,2,n-4)$ (see Figure 11). By Lemma 1.2, $\mu(G_2) \leq \mu(G_1)$. If

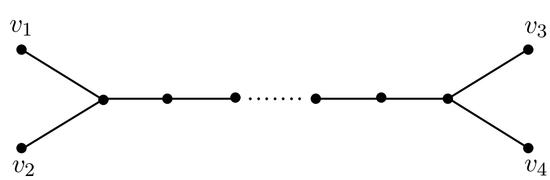

FIGURE 11. The tree $T(2,2,n-4)$

the diameter of G_2 is less than $n-3$, then form a new tree G_3 from G_2, by grafting of edges, such that the diameter of G_3 is $n-3$ and G_3 is not isomorphic to $T(2,2,n-4)$. By Proposition 2.3, $\mu(G_3) \leq \mu(G_2)$. By Proposition 2.1, $\mu(T(2,2,n-4)) < \mu(G_3)$. Now, $C_{3,3}^{n-6}$ is isomorphic to the graph obtained from $T(2,2,n-4)$ by adding two new edges $\{v_1, v_2\}$ and $\{v_3, v_4\}$. Since $\mu(T(2,2,n-4)) = \mu\left(C_{3,3}^{n-6}\right)$ by Theorem 3.4, it follows that $\mu\left(C_{3,3}^{n-6}\right) < \mu(G_1)$. This completes the proof. □

6. Unicyclic Graphs

The algebraic connectivity of unicyclic graphs of order n with fixed girth has been extensively studied by Fallat et al. [2, 3, 4]. Let $\mathcal{U}_n,\ n \geq 3$, denote the class of all unicyclic graphs of order n. Here we discuss the question of extremizing the algebraic connectivity over $\mathcal{U}_n$. For $n = 3$, $\mathcal{U}_n$ contains only one graph. So we assume that $n \geq 4$.

First consider the maximizing case. For the cycle C_n of order n, we have $\mu(C_n) > 1$ if and only if $n \leq 5$, and $\mu(C_n) = 1$ if and only if $n = 6$. Also, a graph in $\mathcal{U}_n$, which is not isomorphic to C_n, has at least one cut-vertex and hence it has algebraic connectivity at most one (Corollary 3.2). Again note that the graph P_n^{n-3} (see Figure 4) is a member of $\mathcal{U}_n$ and $\mu(P_n^{n-3}) = 1$ (Lemma 3.1(ii)). Therefore, if $n \leq 6$, we have the following:
for $n \leq 5$, clearly C_n is the only graph maximizing the algebraic connectivity over $\mathcal{U}_n$ and for $n = 6$, C_6 and P_6^3 are two non-isomorphic graphs in $\mathcal{U}_n$ having algebraic connectivity one.

Now, suppose that $n \geq 6$ and let G be a graph in $\mathcal{U}_n$ with at least one cut-vertex which is not isomorphic to P_n^{n-3}. Let g be the girth of G. If $g = 3$, then $\mu(G) < 1$ by ([2], Theorem 4.14, p.73). If $g = 4, 5$ or 6 and G_1 is the graph in $\mathcal{U}_{g+1}$ obtained by adding one pendant vertex adjacent to a vertex of C_g. Then $\mu(G_1) \approx 0.8299$ if $g = 4$, $\mu(G_1) \approx 0.6972$ if $g = 5$ and $\mu(G_1) \approx 0.5858$ if $g = 6$. Since G is obtained from G_1 by a finite sequence of addition of pendant vertices, by Lemma 1.1, $\mu(G) \leq \mu(G_1) < 1$. For $g \geq 7$, G can be obtained from C_g by a finite sequence of graph operations consisting of addition of pendant vertices. So, by Lemma 1.1, $\mu(G) \leq \mu(C_g) < 1$. Thus, we have the following result.

Theorem 6.1. *The maximum algebraic connectivity over $\mathcal{U}_n$ is uniquely attained by C_n if $n \leq 5$ and uniquely attained by P_n^{n-3} if $n > 6$. When $n = 6$, C_6 and P_6^3 are the only two graphs, up to isomorphism, having the maximum algebraic connectivity over $\mathcal{U}_6$.*

We next consider the minimizing case and prove the following.

Theorem 6.2. *The graph C_3^{n-3} (see Figure 5) uniquely attains the minimum algebraic connectivity over $\mathcal{U}_n$.*

Proof. If $n = 4$, then C_4 and C_3^1 are the only two graphs, up to isomorphism, in $\mathcal{U}_4$ with $\mu(C_3^1) = 1 < \mu(C_4)$. If $n = 5$, then $\mathcal{U}_5$ contains precisely five graphs (up to isomorphism). They are C_3^2, C_5, P_5^2, C_4^1 (adding a pendant vertex to a vertex of C_4) and H (adding two pendant vertices to two distinct vertices of C_3). We have $\mu(C_3^2) \approx 0.5188$, $\mu(H) \approx 0.6972$, $\mu(C_4^1) \approx 0.8299$, $\mu(P_5^2) = 1$ and $\mu(C_5) > 1$. Thus C_3^2 has the minimum algebraic connectivity.

Now, let $n \geq 6$. Let G be a graph in $\mathcal{U}_n$ with at least one pendant vertex which is not isomorphic to C_3^{n-3}. By the same argument used in the proof of Theorem 4.5, we can form a graph G_2 (see Figure 6) from G such that $\mu(C_3^{n-3}) = \mu(G_2) < \mu(G)$. Thus, we only need to show that $\mu(G_2) < \mu(C_n)$. In G_2, it can be verified that $\left\{v_{\lfloor \frac{n}{2} \rfloor}, v_{\lfloor \frac{n}{2} \rfloor+1}\right\}$ is the characteristic edge ([16], Lemma 2.2, p.386). So, in $G_2 - v_{\lfloor \frac{n}{2} \rfloor}$, the component C, consisting

of the path on $\lfloor \frac{n}{2} \rfloor - 1$ vertices is not a Perron component. Therefore,

$$\mu(G_2) < \rho(\widehat{(C)}^{-1}) = 2\left(1 - \cos\frac{\pi}{2(\lfloor \frac{n}{2} \rfloor - 1) + 1}\right) < 2\left(1 - \cos\frac{2\pi}{n}\right) = \mu(C_n).$$

Thus, we have the required result. □

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